JOURNAL OF APPROXIMATION THEORY **89**, 344–371 (1997)
ARTICLE NO. AT963057

Orthogonal Rational Functions and Nested Disks

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Communicated by Hans Wallin

Received June 9, 1995; accepted in revised form March 18, 1996

In Akhiezer's book ["The Classical Moment Problem and Some Related Questions in Analysis," Oliver & Boyd, Edinburgh/London, 1965] the uniqueness of the solution of the Hamburger moment problem, if a solution exists, is related to a theory of nested disks in the complex plane. The purpose of the present paper is to develop a similar nested disk theory for a moment problem that arises in the study of certain orthogonal rational functions. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in the open unit disk in the complex plane, let

$$\mathbb{B}_0 = 1 \qquad \text{and} \qquad \mathbb{B}_n(z) = \prod_{k=0}^{n} \frac{\overline{\alpha_k}}{|\alpha_k|} \frac{\alpha_k - z}{1 - \overline{\alpha_k} z}, \qquad n = 1, 2, ...,$$

($\overline{\alpha_k}/|\alpha_k| = -1$ when $\alpha_k = 0$), and let

$$\mathscr{L} = \text{span}\{\mathbb{B}_n : n = 0, 1, 2, ...\}.$$

0021-9045/97 $25.00

We consider the following "moment" problem:

> Given a positive-definite Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{L} \times \mathscr{L}$, find a non-decreasing function μ on $[-\pi, \pi]$ (or a positive Borel measure μ on $[-\pi, \pi)$) such that
>
> $$\langle f, g \rangle = \int_{-\pi}^{\pi} f(e^{i\theta})\, \overline{g(e^{i\theta})}\, d\mu(\theta) \qquad \text{for} \quad f, g \in \mathscr{L}.$$

In particular we give necessary and sufficient conditions for the uniqueness of the solution in the case that

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty.$$

If this series diverges the solution is always unique.

1. INTRODUCTION

In [2] the uniqueness of the solution of the Hamburger moment problem, if a solution exists, is related to a theory of nested disks in the complex plane. The purpose of the present paper is to develop a similar nested disk theory for a moment problem that arises in the study of certain orthogonal rational functions.

Let

$$T = \{z \in \mathbb{C}: |z| = 1\}, \qquad D = \{z \in \mathbb{C}: |z| < 1\}, \qquad E = \{z \in \mathbb{C}: |z| > 1\}$$

And let α_n, $n = 0, 1, 2, \ldots$ be given points in D with $\alpha_0 = 0$. The Blaschke factors ζ_n are given by

$$\zeta_n(z) = \frac{\overline{\alpha_n}}{|\alpha_n|} \cdot \frac{\alpha_n - z}{1 - \overline{\alpha_n} z}, \qquad n = 0, 1, 2, \ldots,$$

where by convention

$$\frac{\overline{\alpha_n}}{|\alpha_n|} = -1 \qquad \text{when} \quad \alpha_n = 0.$$

The (finite) Blaschke products are

$$\mathbb{B}_n(z) = \prod_{k=1}^{n} \zeta_k(z), \qquad n = 1, 2, \ldots, \qquad \text{and} \qquad \mathbb{B}_0(z) = 1.$$

We define the linear spaces $\mathcal{L}_n$, $n=0, 1, 2, \ldots$ and $\mathcal{L}$ by

$$\mathcal{L}_n=\operatorname{span}\{\mathbb{B}_m\colon m=0, 1, \ldots, n\} \qquad \text{and} \qquad \mathcal{L}=\bigcup_{n=0}^{\infty} \mathcal{L}_n.$$

Clearly $\mathcal{L}_n$ consists of the functions that may be written as

$$\frac{p_n(z)}{\pi_n(z)},$$

where

$$\pi_n(z)=\prod_{k=1}^{n} (1-\overline{\alpha_n} z), \qquad n=1, 2, \ldots, \quad \text{and} \quad \pi_0(z)=1$$

and p_n belongs to $\prod_n$, the set of polynomials of degree at most n. The substar conjugate f_* of a function f is defined as

$$f_*(z)=\overline{f(1/\bar{z})}.$$

For $f \in \mathcal{L}_n \backslash \mathcal{L}_{n-1}$ the superstar conjugate f^* will be

$$f^*(z)=\mathbb{B}_n(z)\, f_*(z).$$

If $f \in \mathcal{L}_0$, then $f^*=f_*$.

The linear spaces $\mathcal{L}_{n*}$, $n=0, 1, 2, \ldots$, and $\mathcal{L}_*$ are defined as

$$\mathcal{L}_{n*}=\{f_*\colon f \in \mathcal{L}_n\} \qquad \text{and} \qquad \mathcal{L}_*=\{f_*\colon f \in \mathcal{L}\}.$$

Then we have

$$\mathcal{L}_{n*}=\operatorname{span}\left\{\frac{1}{\mathbb{B}_m}\colon m=0, 1, \ldots, n\right\}=\operatorname{span}\left\{\frac{1}{\omega_m}\colon m=0, 1, \ldots, n\right\},$$

where

$$\omega_m(z)=\prod_{k=1}^{m} (z-\alpha_k), \qquad \text{and} \qquad \omega_0(z)=1.$$

As in [3] we also put

$$\mathcal{L}_n(\alpha_n)=\{f \in \mathcal{L}_n\colon f(\alpha_n)=0\}, \qquad n=1, 2, \ldots$$

and similarly

$$\mathcal{L}_{n*}(1/\bar{\alpha}_n)=\{f \in \mathcal{L}_{n*}\colon f(1/\overline{\alpha_n})=0\}, \qquad n=1, 2, \ldots.$$

Furthermore we assume that M is a linear functional on $\mathscr{L} + \mathscr{L}_*$ such that for $f \in \mathscr{L}$ we have

$$M(f_*) = \overline{M(f)}, \qquad \text{and} \qquad M(ff_*) > 0 \qquad \text{if} \quad f \neq 0.$$

Then this also holds for $f \in \mathscr{L} + \mathscr{L}_*$. The functional M induces an inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{L} \times \mathscr{L}$ by

$$\langle f, g \rangle = M(fg_*), \qquad f, g \in \mathscr{L}.$$

Note that $\mathscr{L}\mathscr{L}_* = \mathscr{L} + \mathscr{L}_*$, as can be seen by partial fraction decomposition. Also for $f, g \in \mathscr{L}_*$ we may define $\langle f, g \rangle = M(fg_*)$. Then we get

$$\langle f, g \rangle = \langle g_*, f_* \rangle \qquad \text{for} \quad f, g \in \mathscr{L}.$$

As $\overline{\langle g, f \rangle} = \overline{M(gf_*)} = M(fg_*) = \langle f, g \rangle$ for $f, g \in \mathscr{L}$ and $\langle f, f \rangle = M(ff_*) > 0$ for $f \in \mathscr{L}$, $f \neq 0$, the inner product is Hermitian and positive-definite on $\mathscr{L} \times \mathscr{L}$.

In this paper we develop a nested disk theory in connection to the following "moment" problem:

> Given the inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{L} \times \mathscr{L}$ (or the linear functional M on $\mathscr{L} + \mathscr{L}_*$), find a non-decreasing function μ on $[-\pi, \pi]$ (or a positive Borel measure μ on $[-\pi, \pi)$) such that

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(e^{i\theta})\, \overline{g(e^{i\theta})}\, d\mu(\theta) \qquad \text{for} \quad f, g \in \mathscr{L}$$

$$\left(\text{or} \quad M(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\, d\mu(\theta) \qquad \text{for} \quad f \in \mathscr{L} + \mathscr{L}_*\right).$$

In particular we give necessary and sufficient conditions for the uniqueness of the solution in the case that

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty.$$

If this series diverges the solution is always unique. This is a consequence of the "closure criterion" discussed in Addendum A.2 of [1]. Two non-decreasing functions which are solutions of the moment problem such that their difference is a constant at all the points at which it is continuous are considered to be the same solution of the moment problem.

2. ORTHOGONAL RATIONAL FUNCTIONS

In our approach orthogonal rational functions will play an important role. Let the sequence $\{\phi_n\}_{n=0}^{\infty}$ in $\mathcal{L}$ be obtained by orthonormalization of the sequence $\{\mathbb{B}_n\}_{n=0}^{\infty}$ with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{L} \times \mathcal{L}$, i.e.,

$$\phi_n \in \mathcal{L}_n \qquad \text{and} \qquad \langle \phi_n, \phi_n \rangle = 1, \qquad n = 0, 1, 2, \ldots$$

and

$$\langle f, \phi_n \rangle = 0 \qquad \text{for} \quad f \in \mathcal{L}_{n-1}, \qquad n = 1, 2, \ldots.$$

Such orthogonal rational systems were also considered by Djrbashian [9]. It follows easily that

$$\langle f, \phi_n^* \rangle = 0 \qquad \text{for} \quad f \in \mathcal{L}_n(\alpha_n), \qquad n = 1, 2, \ldots,$$

because $\mathbb{B}_n f_* \in \mathcal{L}_{n-1}$ for such f. Each ϕ_n can be written as

$$\phi_n(z) = \sum_{k=0}^{n} b_k^{(n)} \mathbb{B}_k(z).$$

Here the non-zero number $b_n^{(n)}$ is called the leading coefficient of ϕ_n. We assume that the ϕ_n are chosen such that $b_n^{(n)} > 0$ and we write $\kappa_n = b_n^{(n)}$. It is easily shown that

$$\kappa_n = \overline{\phi_n^*(\alpha_n)} = \phi_n^*(\alpha_n).$$

Using the uniqueness of the reproducing kernel

$$\sum_{k=0}^{n} \phi_k(z) \overline{\phi_k(w)}$$

for the inner product space $\mathcal{L}_n$ one can show, see for instance [3] or [9], that the following Christoffel–Darboux formula holds

$$\sum_{k=0}^{n-1} \phi_k(z) \overline{\phi_k(w)} = \frac{\phi_n^*(z) \overline{\phi_n^*(w)} - \phi_n(z) \overline{\phi_n(w)}}{1 - \zeta_n(z) \overline{\zeta_n(w)}}, \tag{2.1}$$

and equivalently

$$\sum_{k=0}^{n} \phi_k(z) \overline{\phi_k(w)} = \frac{\phi_n^*(z) \overline{\phi_n^*(w)} - \zeta_n(z) \overline{\zeta_n(w)} \phi_n(z) \overline{\phi_n(w)}}{1 - \zeta_n(z) \overline{\zeta_n(w)}}. \tag{2.2}$$

The ϕ_n and ϕ_n^* satisfy the recurrence relations

$$\phi_n(z) = \varepsilon_n \frac{z - \alpha_{n-1}}{1 - \overline{\alpha_n} z} \frac{\kappa_n}{\kappa_{n-1}} \phi_{n-1}(z) + \delta_n \frac{1 - \overline{\alpha_{n-1}} z}{1 - \overline{\alpha_n} z} \frac{\kappa_n}{\kappa_{n-1}} \phi_{n-1}^*(z),$$
$$n = 1, 2, \ldots \tag{2.3}$$

and (superstar conjugation)

$$\phi_n^*(z) = -\frac{\overline{\alpha_n}}{|\alpha_n|} \overline{\delta_n} \frac{z - \alpha_{n-1}}{1 - \overline{\alpha_n} z} \frac{\kappa_n}{\kappa_{n-1}} \phi_{n-1}(z)$$
$$- \frac{\overline{\alpha_n}}{|\alpha_n|} \overline{\varepsilon_n} \frac{1 - \overline{\alpha_{n-1}} z}{1 - \overline{\alpha_n} z} \frac{\kappa_n}{\kappa_{n-1}} \phi_{n-1}^*(z), \qquad n = 1, 2, \ldots \tag{2.4}$$

with $\phi_0 = \phi_0^* = \kappa_0$. Here

$$\varepsilon_n = -\frac{\overline{\alpha_n}}{|\alpha_n|} \frac{1 - \overline{\alpha_{n-1}} \alpha_n}{1 - |\alpha_{n-1}|^2} \frac{\overline{\phi_n^*(\alpha_{n-1})}}{\kappa_n}, \tag{2.5}$$

$$\delta_n = \frac{1 - \alpha_{n-1} \overline{\alpha_n}}{1 - |\alpha_{n-1}|^2} \frac{\phi_n(\alpha_{n-1})}{\kappa_n}. \tag{2.6}$$

It follows from the Christoffel–Darboux formula (2.1) with $z = w = \alpha_{n-1}$ that $\varepsilon_n \neq 0$. A proof of (2.3) and (2.4) can be found in [3] or in [4], but (2.3) and (2.4) also may be derived from the superstar conjugates with respect to w and with repect to z and w of the Christoffel–Darboux formula. See also [9]. We mention another consequence of the Christoffel–Darboux formula. Taking the superstar conjugate of (2.1) with respect to z and w and writing

$$\mathbb{B}_{n \setminus k} = \mathbb{B}_n / \mathbb{B}_k, \qquad k = 0, 1, \ldots, n; \quad n = 0, 1, \ldots$$

we obtain

$$\frac{\phi_n^*(z) \overline{\phi_n^*(w)} - \phi_n(z) \overline{\phi_n(w)}}{1 - \zeta_n(z) \overline{\zeta_n(w)}} = \sum_{k=0}^{n-1} \mathbb{B}_{(n-1) \setminus k}(z) \overline{\mathbb{B}_{(n-1) \setminus k}(w)} \phi_k^*(z) \overline{\phi_k^*(w)}. \tag{2.7}$$

For $z = w = \alpha_{n-1}$ this gives

$$|\phi_n^*(\alpha_{n-1})|^2 - |\phi_n(\alpha_{n-1})|^2 = |\phi_{n-1}^*(\alpha_{n-1})|^2 [1 - |\zeta_n(\alpha_{n-1})|^2]$$
$$= \kappa_{n-1}^2 \frac{(1 - |\alpha_n|^2)(1 - |\alpha_{n-1}|^2)}{|1 - \overline{\alpha_n} \alpha_{n-1}|^2}.$$

Together with (2.5) and (2.6) this leads to

$$|\varepsilon_n|^2 - |\delta_n|^2 = \frac{\kappa_{n-1}^2}{\kappa_n^2} \frac{1-|\alpha_n|^2}{1-|\alpha_{n-1}|^2}. \tag{2.8}$$

In particular this implies that

$$|\varepsilon_n| > |\delta_n|. \tag{2.9}$$

A different proof of (2.8) can be found in [6].

3. ASSOCIATED FUNCTIONS

Next to the orthogonal functions ϕ_n we consider the associated functions ψ_n defined by

$$\psi_0(z) = -\frac{1}{\kappa_0}, \qquad (\psi_0(z) = -M(\phi_0)),$$

and

$$\psi_n(z) = M(D(t, z)[\phi_n(z) - \phi_n(t)]), \qquad n = 1, 2, \ldots.$$

Here M is acting on t and

$$D(t, z) = \frac{t+z}{t-z}.$$

Obviously $\psi_n \in \mathscr{L}_n$ for $n = 0, 1, 2, \ldots$. It is shown in [8] that

$$\psi_n(z) = M\left(D(t, z)\left[\phi_n(z) - \frac{f(t)}{f(z)}\phi_n(t)\right]\right)$$
$$\text{for} \quad f \in \mathscr{L}_{(n-1)*}, \quad f \not\equiv 0, \quad n = 1, 2, \ldots. \tag{3.1}$$

For the superstar conjugates of the ψ_n we have

$$\psi_0^*(z) = -\frac{1}{\kappa_0}$$

and

$$\psi_n^*(z) = M\left(D(t, z)\left[\frac{\mathbb{B}_n(z)}{\mathbb{B}_n(t)}\phi_n^*(t) - \phi_n^*(z)\right]\right), \qquad n = 1, 2, \ldots. \tag{3.2}$$

It is shown in [8] that we also have

$$\psi_n^*(z) = M\left(D(t, z)\left[\frac{f(t)}{f(z)}\,\phi_n^*(t) - \phi_n^*(z) \right] \right)$$
$$\text{for} \quad f \in \mathscr{L}_{n*}(1/\overline{\alpha_n}), \quad f \not\equiv 0, \quad n = 1, 2, \ldots. \tag{3.3}$$

The functions ψ_n and ψ_n^* satisfy the recurrences

$$\psi_n(z) = \varepsilon_n \frac{z - \alpha_{n-1}}{1 - \overline{\alpha_n} z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}(z) - \delta_n \frac{1 - \overline{\alpha_{n-1}} z}{1 - \overline{\alpha_n} z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}^*(z),$$
$$n = 1, 2, \ldots \tag{3.4}$$

and (superstar conjugation)

$$\psi_n^*(z) = \frac{\overline{\alpha_n}}{|\alpha_n|}\,\overline{\delta_n}\,\frac{z - \alpha_{n-1}}{1 - \overline{\alpha_n} z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}(z)$$
$$- \frac{\overline{\alpha_n}}{|\alpha_n|}\,\overline{\varepsilon_n}\,\frac{1 - \overline{\alpha_{n-1}} z}{1 - \overline{\alpha_n} z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}^*(z), \qquad n = 1, 2, \ldots. \tag{3.5}$$

A proof of these recurrence formulas can be found in [3]. Another proof is given in [8]. The pair $(\psi_n, -\psi_n^*)$ satisfies the same recurrence as the pair (ϕ_n, ϕ_n^*). The initial values are $(\phi_0, \phi_0^*) = \kappa_0(1, 1)$ and $(\psi_0, -\psi_0^*) = (-1/\kappa_0)(1, -1)$.

4. ANALOGUES OF THE LIOUVILLE–OSTROGRADSKII FORMULA (DETERMINANT FORMULA) AND GREEN'S FORMULA

In the previous section we have seen that the pairs $(\phi_n(z), \phi_n^*(z))$ and $(\psi_n(z), -\psi_n^*(z))$ satisfy the recurrence

$$\frac{\kappa_{n-1}}{\kappa_n} X_n(z) = \varepsilon_n A_n(z)\, X_{n-1}(z) + \delta_n B_n(z)\, X_{n-1}^\dagger(z), \qquad n = 1, 2, \ldots, \tag{4.1a}$$

$$\frac{\kappa_{n-1}}{\kappa_n} X_n^\dagger(z) = \tau_n [\overline{\delta_n} A_n(z)\, X_{n-1}(z) + \overline{\varepsilon_n} B_n(z)\, X_{n-1}^\dagger(z)], \qquad n = 1, 2, \ldots, \tag{4.1b}$$

where

$$\tau_n = -\frac{\overline{\alpha_n}}{|\alpha_n|}, \qquad A_n(z) = \frac{z - \alpha_{n-1}}{1 - \overline{\alpha_n} z}, \qquad B_n(z) = \frac{1 - \overline{\alpha_{n-1}} z}{1 - \overline{\alpha_n} z}, \qquad n = 1, 2, \ldots.$$

Suppose that the pair $(x_n(z), x_n^\dagger(z))$ satisfies (4.1) and suppose that the pair $(y_n(w), y_n^\dagger(w))$ satisfies (4.1) with z replaced by w. Put

$$G_n(z, w) = x_n^\dagger(z)\, y_n(w) - x_n(z)\, y_n^\dagger(w), \qquad n = 0, 1, 2, \dots.$$

Then

$$\begin{aligned}
\frac{\kappa_{n-1}^2}{\kappa_n^2} G_n(z, w) &= \tau_n[\overline{\delta_n} A_n(z)\, x_{n-1}(z) + \overline{\varepsilon_n} B_n(z)\, x_{n-1}^\dagger(z)] \\
&\quad \cdot [\varepsilon_n A_n(w)\, y_{n-1}(w) + \delta_n B_n(w)\, y_{n-1}^\dagger(w)] \\
&\quad - [\varepsilon_n A_n(z)\, x_{n-1}(z) + \delta_n B_n(z)\, x_{n-1}^\dagger(z)] \\
&\quad \cdot \tau_n[\overline{\delta_n} A_n(w)\, y_{n-1}(w) + \overline{\varepsilon_n} B_n(w)\, y_{n-1}^\dagger(w)] \\
&= \tau_n[|\delta_n|^2 - |\varepsilon_n|^2]\, A_n(z)\, B_n(w)\, x_{n-1}(z)\, y_{n-1}^\dagger(w) \\
&\quad + \tau_n[|\varepsilon_n|^2 - |\delta_n|^2]\, A_n(w)\, B_n(z)\, x_{n-1}^\dagger(z)\, y_{n-1}(w) \\
&= \tau_n[|\varepsilon_n|^2 - |\delta_n|^2]\{A_n(w)\, B_n(z)\, x_{n-1}^\dagger(z)\, y_{n-1}(w) \\
&\quad - A_n(z)\, B_n(w)\, x_{n-1}(z)\, y_{n-1}^\dagger(w)\} \\
&= \tau_n[|\varepsilon_n|^2 - |\delta_n|^2]\, A_n(w)\, B_n(z) \\
&\quad \cdot \left[x_{n-1}^\dagger(z)\, y_{n-1}(w) - x_{n-1}(z)\, y_{n-1}^\dagger(w) \right. \\
&\quad \left. + \left\{1 - \frac{A_n(z)\, B_n(w)}{A_n(w)\, B_n(z)}\right\} x_{n-1}(z)\, y_{n-1}^\dagger(w) \right].
\end{aligned}$$

Here

$$1 - \frac{A_n(z)\, B_n(w)}{A_n(w)\, B_n(z)} = \frac{(1 - |\alpha_{n-1}|^2)(z - w)}{(\alpha_{n-1} - w)(1 - \overline{\alpha_{n-1}} z)} = 1 - \frac{\zeta_{n-1}(z)}{\zeta_{n-1}(w)}.$$

With

$$\frac{c_{n-1}}{c_n} = \tau_n[|\varepsilon_n|^2 - |\delta_n|^2]\, A_n(w)\, B_n(z), \qquad n = 1, 2, \dots \qquad \text{and} \qquad c_0 = 1$$

we get

$$\begin{aligned}
\frac{1}{c_n} &= \prod_{k=1}^{n} [|\varepsilon_k|^2 - |\delta_k|^2] \cdot \prod_{k=1}^{n} \left(-\frac{\overline{a_k}}{|\alpha_k|}\right) \frac{w - \alpha_k}{1 - \overline{\alpha_k} w} \cdot \prod_{k=1}^{n} \frac{w - \alpha_{k-1}}{w - \alpha_k} \\
&\quad \cdot \prod_{k=1}^{n} \frac{1 - \overline{\alpha_{k-1}} z}{1 - \overline{\alpha_k} z}.
\end{aligned}$$

Using (2.8) this gives

$$c_n = \frac{\kappa_n^2}{\kappa_0^2} \frac{1-\overline{\alpha_n} z}{1-|\alpha_n|^2} \frac{w-\alpha_n}{w} \frac{1}{\mathbb{B}_n(w)}. \tag{4.2}$$

Moreover

$$c_{n-1} \left\{ 1 - \frac{A_n(z)\, B_n(w)}{A_n(w)\, B_n(z)} \right\} = -\frac{\kappa_{n-1}^2}{\kappa_0^2} \frac{z-w}{w\mathbb{B}_{n-1}(w)}.$$

Hence

$$\frac{c_n}{\kappa_n^2} G_n(z, w) - \frac{c_{n-1}}{\kappa_{n-1}^2} G_{n-1}(z, w) = -\frac{1}{\kappa_0^2} \frac{z-w}{w\mathbb{B}_{n-1}(w)} x_{n-1}(z)\, y_{n-1}^\dagger(w),$$

and summation gives ($c_0 = 1$)

$$\frac{\kappa_0^2}{\kappa_n^2} c_n G_n(z, w) - G_0(z, w) = -\frac{z-w}{w} \sum_{k=1}^{n} \frac{x_{k-1}(z)\, y_{k-1}^\dagger(w)}{\mathbb{B}_{k-1}(w)},$$

and by (4.2)

$$\frac{1-\overline{\alpha_n} z}{1-|\alpha_n|^2} \frac{w-\alpha_n}{z-w} \frac{1}{\mathbb{B}_n(w)} G_n(z, w) - \frac{w}{z-w} G_0(z, w) = -\sum_{k=0}^{n-1} \frac{x_k(z)\, y_k^\dagger(w)}{\mathbb{B}_k(w)},$$

so

$$\begin{aligned} &\frac{x_n^\dagger(z)\, y_n(w) - x_n(z)\, y_n^\dagger(w)}{1-(\zeta_n(z)/\zeta_n(w))} - \mathbb{B}_n(w) \frac{x_0^\dagger(z)\, y_0(w) - x_0(z)\, y_0^\dagger(w)}{1-(\zeta_0(z)/\zeta_0(w))} \\ &\quad = -\sum_{k=0}^{n-1} x_k(z)\, \mathbb{B}_{n\backslash k}(w)\, y_k^\dagger(w). \end{aligned}$$

For $z = w$ we get the determinant formula

$$x_n^\dagger(z)\, y_n(z) - x_n(z)\, y_n^\dagger(z) = \frac{1-|\alpha_n|^2}{1-\overline{\alpha_n} z} \frac{z\mathbb{B}_n(z)}{z-\alpha_n} (x_0^\dagger(z)\, y_0(z) - x_0(z)\, y_0^\dagger(z)).$$

In particular for

$$x_n(z) = \phi_n(z), \qquad x_n^\dagger(z) = \phi_n^*(z), \qquad y_n(z) = \psi_n(z), \qquad y_n^\dagger(z) = -\psi_n^*(z)$$

we obtain the analogue of the Liouville–Ostrogradskii formula

$$\phi_n^*(z)\, \psi_n(z) + \phi_n(z)\, \psi_n^*(z) = \frac{1-|\alpha_n|^2}{1-\overline{\alpha_n} z} \frac{-2z\mathbb{B}_n(z)}{z-\alpha_n} \tag{4.3}$$

as a special case of the determinant formula. Of course formula (4.3) may be derived more easily. The formula is also proved in [6].

The analogue of Green's formula is derived in a similar way. Put

$$F_n(z, w) = x_n^\dagger(z)\,\overline{y_n^\dagger(w)} - x_n(z)\,\overline{y_n(w)}, \qquad n = 0, 1, 2, \ldots.$$

This time we obtain from (4.1)

$$\frac{\kappa_{n-1}^2}{\kappa_n^2} F_n(z, w) = [\,|\varepsilon_n|^2 - |\delta_n|^2]\, B_n(z)\,\overline{B_n(w)}$$
$$\cdot \left[F_{n-1}(z, w) + \left\{ 1 - \frac{A_n(z)\,\overline{A_n(w)}}{B_n(z)\,\overline{B_n(w)}} \right\} x_{n-1}(z)\,\overline{y_{n-1}(w)} \right],$$

where now

$$1 - \frac{A_n(z)\,\overline{A_n(w)}}{B_n(z)\,\overline{B_n(w)}} = \frac{(1 - |\alpha_{n-1}|^2)(1 - z\bar{w})}{(1 - \overline{\alpha_{n-1}}\,z)(1 - \alpha_{n-1}\bar{w})} = 1 - \zeta_{n-1}(z)\,\overline{\zeta_{n-1}(w)}.$$

If $c_0 = 1$ and

$$\frac{c_{n-1}}{c_n} = [\,|\varepsilon_n|^2 - |\delta_n|^2]\, B_n(z)\,\overline{B_n(w)}, \qquad n = 1, 2, \ldots,$$

then

$$c_n = \frac{\kappa_n^2}{\kappa_0^2} \frac{(1 - \overline{\alpha_n}z)(1 - \alpha_n\bar{w})}{1 - |\alpha_n|^2} = \frac{\kappa_n^2}{\kappa_0^2} \frac{1 - z\bar{w}}{1 - \zeta_n(z)\,\overline{\zeta_n(w)}}$$

and

$$c_{n-1} \left\{ 1 - \frac{A_n(z)\,\overline{A_n(w)}}{B_n(z)\,\overline{B_n(w)}} \right\} = \frac{\kappa_{n-1}^2}{\kappa_0^2} (1 - z\bar{w}).$$

Thus we obtain

$$\frac{c_n}{\kappa_n^2} F_n(z, w) - \frac{c_{n-1}}{\kappa_{n-1}^2} F_{n-1}(z, w) = \frac{1}{\kappa_0^2} (1 - z\bar{w})\, x_{n-1}(z)\,\overline{y_{n-1}(w)}$$

which leads to

$$\frac{c_n}{\kappa_n^2} F_n(z, w) - \frac{c_0}{\kappa_0^2} F_0(z, w) = \frac{1}{\kappa_0^2} (1 - z\bar{w}) \sum_{k=0}^{n-1} x_k(z)\,\overline{y_k(w)},$$

and using the expression for c_n,

$$\frac{1-z\bar{w}}{1-\zeta_n(z)\,\overline{\zeta_n(w)}}\,F_n(z,w)-F_0(z,w)=(1-z\bar{w})\sum_{k=0}^{n-1}x_k(z)\,\overline{y_k(w)},$$

so

$$\frac{x_n^\dagger(z)\,\overline{y_n^\dagger(w)}-x_n(z)\,\overline{y_n(w)}}{1-\zeta_n(z)\,\overline{\zeta_n(w)}}-\frac{x_0^\dagger(z)\,\overline{y_0^\dagger(w)}-x_0(z)\,\overline{y_0(w)}}{1-z\bar{w}}=\sum_{k=0}^{n-1}x_k(z)\,\overline{y_k(w)}, \tag{4.4}$$

the analogue of Green's formula. Notice that $1-\zeta_0(z)\,\overline{\zeta_0(w)}=1-z\bar{w}$.

We only mention the following special cases of Green's formula. For

$$x_n(z)=\phi_n(z),\qquad x_n^\dagger(z)=\phi_n^*(z),\qquad y_n(w)=\psi_n(w),\qquad y_n^\dagger(w)=-\psi_n^*(w)$$

we get

$$\frac{\phi_n^*(z)\,\overline{\psi_n^*(w)}+\phi_n(z)\,\overline{\psi_n(w)}}{1-\zeta_n(z)\,\overline{\zeta_n(w)}}+\frac{2}{1-z\bar{w}}=-\sum_{k=0}^{n-1}\phi_k(z)\,\overline{\psi_k(w)}. \tag{4.5}$$

For

$$x_n(z)=\psi_n(z),\qquad x_n^\dagger(z)=-\psi_n^*(z),\qquad y_n(w)=\psi_n(w),\qquad y_n^\dagger(w)=-\psi_n^*(w)$$

we get a "Christoffel–Darboux" formula for the associated functions

$$\frac{\psi_n^*(z)\,\overline{\psi_n^*(w)}-\psi_n(z)\,\overline{\psi_n(w)}}{1-\zeta_n(z)\,\overline{\zeta_n(w)}}=\sum_{k=0}^{n-1}\psi_k(z)\,\overline{\psi_k(w)}, \tag{4.6}$$

and if $z=w$

$$\frac{|\psi_n^*(z)|^2-|\psi_n(z)|^2}{1-|\zeta_n(z)|^2}=\sum_{k=0}^{n-1}|\psi_k(z)|^2. \tag{4.7}$$

The superstar conjugate of (4.7) reads

$$\frac{|\psi_n^*(z)|^2-|\psi_n(z)|^2}{1-|\zeta_n(z)|^2}=\sum_{k=0}^{n-1}|\mathbb{B}_{(n-1)\backslash k}(z)|^2\,|\psi_k^*(z)|^2. \tag{4.8}$$

5. PARA-ORTHOGONAL FUNCTIONS AND QUADRATURE FORMULAS

It follows easily from the Christoffel–Darboux formula (2.1) that the zeros of ϕ_n are in D and that the zeros of ϕ_n^* are in E. Moreover we have

$|\phi_n(z)| < |\phi_n^*(z)|$ for $z \in D$ and $|\phi_n(z)| > |\phi_n^*(z)|$ for $z \in E$. As we intend to give quadrature formulas with nodes in T we consider the functions

$$Q_n(z, w) = \phi_n(z) + w\phi_n^*(z), \qquad n = 0, 1, 2, ... \tag{5.1}$$

with $w \in T$ arbitrary. Clearly the zeros $z_1, ..., z_n$ of $Q_n(z, w)$ are all in T and it is easy to show that they are simple. See [3]. Of course the zeros z_j depend on n and w. Since

$$Q_n(z, w) \perp \mathscr{L}_{n-1} \cap \mathscr{L}_n(\alpha_n), \qquad n = 1, 2, ...$$

and

$$\langle Q_n(z, w), 1 \rangle \neq 0 \qquad \text{and} \qquad \langle Q_n(z, w), \mathbb{B}_n(z) \rangle \neq 0, \qquad n = 1, 2, ...,$$

where the inner product acts on z, the sequence is called para-orthogonal. As

$$Q_n^*(z, w) = \bar{w} Q_n(z, w),$$

superstar conjugation with respect to z, the Q_n are called $\bar{w}$-invariant. Notice that the above orthogonality remains valid if for each n we take for w a fixed w_n in T. If

$$\Lambda_{n, i}(z) = \frac{1 - \overline{\alpha_n} z}{1 - \overline{\alpha_n} z_i} \frac{Q_n(z, w)}{(z - z_i)\, Q_n'(z_i, w)}, \qquad i = 1, ..., n, \tag{5.2}$$

where the prime means differentiation with respect to z, then $\Lambda_{n, i} \in \mathscr{L}_{n-1}$ and we have the quadrature formula (see [3])

$$M(R) = \sum_{j=1}^{n} \lambda_{n, j} R(z_j) \qquad \text{for} \quad R \in \mathscr{L}_{(n-1)*} + \mathscr{L}_{n-1}, \tag{5.3}$$

with $\lambda_{n, j} = M(\Lambda_{n, j}) > 0$ for $j = 1, ..., n$.

Let us assume now that $z_j = e^{i\theta_j}$, $j = 1, 2, ..., n$, with

$$-\pi \leqslant \theta_1 < \theta_2 < \cdots < \theta_n < \pi.$$

Then, using the functions μ_n given by

$$\mu_n(\theta) = \begin{cases} 0 & \text{if} \quad -\pi \leqslant \theta \leqslant \theta_1, \\ \sum_{j=1}^{k} \lambda_{n, j} & \text{if} \quad \theta_k < \theta \leqslant \theta_{k+1}, \qquad k = 1, ..., n-1, \\ M(1) & \text{if} \quad \theta_n < \theta \leqslant \pi \end{cases}$$

(or using the measures $\mu_n = \sum_{j=1}^{n} \lambda_{n,j} \delta_{\theta_j}$, where δ_{θ_j} is the translated Dirac measure), one obtains from Helly's theorems (or from the weak* compactness of the 1-ball in the dual space of the Banach space $C(T)$) that the moment problem has a solution, say μ. So there is a non-decreasing function (or a positive Borel measure) μ such that

$$M(R) = \int_{-\pi}^{\pi} R(e^{i\theta})\, d\mu(\theta) \qquad \text{for} \quad R \in \mathscr{L}_* + \mathscr{L}. \tag{5.4}$$

It follows from the fact that the inner product is positive definite that the solutions μ must have infinitely many points of increase (or must be measures with infinite support). The proof is given in Section 7.

Now let

$$F_\mu(z) = \int_{-\pi}^{\pi} \frac{t+z}{t-z}\, d\mu(\theta) \qquad (t = e^{i\theta}) \tag{5.5}$$

and

$$R_n(z, w) = \int_{-\pi}^{\pi} \frac{t+z}{t-z}\, d\mu_n(\theta) = \sum_{j=1}^{n} \lambda_{n,j} \frac{z_j + z}{z_j - z}.$$

Then $R_n(z, w)$ can be written as

$$R_n(z, w) = \frac{P_n(z, w)}{Q_n(z, w)} \qquad \text{with} \quad P_n(z, w) \in \mathscr{L}_n.$$

It is shown in [8] that

$$P_n(z, w) = \psi_n(z) - w\psi_n^*(z), \qquad n = 1, 2, \ldots. \tag{5.6}$$

In [5] a formula like (5.6) was obtained only in the "cyclic" situation, i.e., in the case of a finite number of points α_n repeated in cyclic order.

From the partial fraction decomposition

$$R_n(z, w) = \sum_{j=1}^{n} \lambda_{n,j} \frac{z_j + z}{z_j - z}$$

it follows that

$$\lambda_{n,k} = -\frac{1}{2z_k} \frac{P_n(z_k, w)}{Q_n'(z_k, w)}, \qquad k = 1, \ldots, n. \tag{5.7}$$

Using the determinant formula it can be shown that (see [8])

$$\lambda_{n,j} = \frac{1}{\sum_{k=0}^{n-1} |\phi_k(z_j)|^2}, \qquad j = 1, \ldots, n; \quad n \in \mathbb{N}. \tag{5.8}$$

It is also shown in [8] that $R_n(z, w)$ is a "Padé-type" approximant to F_μ in the following sense. There are functions h_0 and h_∞, both analytic in $D \cup E$ with $\lim_{z\to 0} h_0(z) = 0$ and $\lim_{z\to\infty} h_\infty(z) = 0$ such that

$$F_\mu(z) - R_n(z, w) = \mathbb{B}_{n-1}(z)\, h_0(z)$$

and

$$F_\mu(z) - R_n(z, w) = \frac{1}{\mathbb{B}_{n-1}(z)}\, h_\infty(z).$$

Of course the functions h_0 and h_∞ depend on the parameter w. The error is given by

$$F_\mu(z) - R_n(z, w) = \frac{1}{f(z)\, Q_n(z, w)} \int_{-\pi}^{\pi} D(t, z)\, f(t)\, Q_n(t, w)\, d\mu(\theta),$$

where $f \in \mathcal{L}_{(n-1)*} \cap \mathcal{L}_{n*}(1/\overline{\alpha_n})$, $f \not\equiv 0$. See also [7].

6. NESTED DISKS

Let

$$D_0 = \{z \in D\colon z \neq \alpha_j, j = 0, 1, 2, ...\} \quad \text{and} \quad E_0 = \{z \in E\colon z \neq 1/\overline{\alpha_j}, j = 1, 2, ...\}.$$

For fixed $z \in D_0 \cup E_0$ the values of

$$s = R_n(z, w) = \frac{\psi_n(z) - w\psi_n^*(z)}{\phi_n(z) + w\phi_n^*(z)}$$

describe a circle, say $K_n(z)$, if w varies in T. Indeed, by the Christoffel–Darboux formula (2.1) and formula (4.7) we have

$$0 < \frac{||\psi_n(z)| - |\psi_n^*(z)||}{|\phi_n(z)| + |\phi_n^*(z)|} \leqslant |s| \leqslant \frac{|\psi_n(z)| + |\psi_n^*(z)|}{||\phi_n(z)| - |\phi_n^*(z)||} < \infty.$$

As $w \in T$, the equation of $K_n(z)$ is

$$|\psi_n^*(z) + s\phi_n^*(z)| = |\psi_n(z) - s\phi_n(z)|. \tag{6.1}$$

Since the pairs $(\phi_n(z), \phi_n^*(z))$ and $(\psi_n(z), -\psi_n^*(z))$ are (independent) solutions of the recurrency (4.1) also the pair $(\psi_n(z) - s\phi_n(z),$

$-\psi_n^*(z)-s\phi_n^*(z))$ is a solution of (4.1). Hence by the analogue of Green's formula (4.4) we have for the given z

$$\frac{|\psi_n^*(z)+s\phi_n^*(z)|^2-|\psi_n(z)-s\phi_n(z)|^2}{1-|\zeta_n(z)|^2}-\frac{|(1/\kappa_0)-s\kappa_0|^2-|(1/\kappa_0)+s\kappa_0|^2}{1-|z|^2}$$
$$=\sum_{k=0}^{n-1}|\psi_k(z)-s\phi_k(z)|^2.$$

Since the first term on the left-hand side of this equation is zero, the equation of the circle $K_n(z)$ is

$$\sum_{k=0}^{n-1}|\psi_k(z)-s\phi_k(z)|^2=\frac{2(s+\bar{s})}{1-|z|^2}. \tag{6.2}$$

Clearly the circular disk $\Delta_n(z)$ corresponding to $K_n(z)$ is given by

$$\sum_{k=0}^{n-1}|\psi_k(z)-s\phi_k(z)|^2\leqslant\frac{2(s+\bar{s})}{1-|z|^2}. \tag{6.3}$$

It follows directly from (6.2) that

$$K_n(z)\subset\{s\in\mathbb{C}\colon \Re s>0\}\qquad\text{if}\quad z\in D_0$$

and

$$K_n(z)\subset\{s\in\mathbb{C}\colon \Re s<0\}\qquad\text{if}\quad z\in E_0.$$

Indeed, if $s\in\Delta_n(z)$ and $\Re s=0$, then $\psi_0(z)-s\phi_0(z)=0$, so $s=-1/\kappa_0^2$. A contradiction. The centre and the radius of $K_n(z)$ follow easily from (6.1). We have

$$\text{centre}=-\frac{\psi_n^*(z)\,\overline{\phi_n^*(z)}+\psi_n(z)\,\overline{\phi_n(z)}}{|\phi_n^*(z)|^2-|\phi_n(z)|^2},$$

$$\text{radius}=\left|\frac{\psi_n^*(z)\,\phi_n(z)+\phi_n^*(z)\,\psi_n(z)}{|\phi_n^*(z)|^2-|\phi_n(z)|^2}\right|.$$

Using (4.5) and (2.1) with $z=w$ we get

$$\text{centre}=\frac{2/(1-|z|^2)+\sum_{k=0}^{n-1}\psi_k(z)\,\overline{\phi_k(z)}}{\sum_{k=0}^{n-1}|\phi_k(z)|^2}.$$

Using (4.3) and (2.1) with $z=w$ we get

$$\begin{aligned}\text{radius} &= \left|\frac{(1-|\alpha_n|^2)/(1-\overline{\alpha_n}z)\cdot(-2z\mathbb{B}_n(z))/(z-\alpha_n)}{(1-|\zeta_n(z)|^2)\cdot\sum_{k=0}^{n-1}|\phi_k(z)|^2}\right| \\ &= \frac{2\,|z|}{|1-|z|^2|}\cdot\frac{|\mathbb{B}_{n-1}(z)|}{\sum_{k=0}^{n-1}|\phi_k(z)|^2}. \end{aligned} \tag{6.4}$$

Formula (6.3) implies that the $\Delta_n(z)$ are nested disks,

$$\Delta_n(z)\supset\Delta_{n+1}(z), \qquad n=1,2,....$$

Moreover, by (6.2), the circles $K_n(z)$ and $K_{n+1}(z)$ touch if z is not a zero of ϕ_n or if both $\phi_n(z)$ and $\psi_n(z)$ are zero.

The intersection of the disks $\Delta_n(z)$ is denoted as $\Delta_\infty(z)$. Clearly $\Delta_\infty(z)$ is a circular disk (with a positive radius) or $\Delta_\infty(z)$ is a point. The limiting circle, which may reduce to a point, is denoted as $K_\infty(z)$. The inequality for $\Delta_\infty(z)$ is

$$\sum_{k=0}^{\infty}|\psi_k(z)-s\phi_k(z)|^2\leqslant\frac{2(s+\bar{s})}{1-|z|^2}. \tag{6.5}$$

As we have nested disks, (6.4) implies that the sequence

$$\left(\frac{|\mathbb{B}_n(z)|}{\sum_{k=0}^{n}|\phi_k(z)|^2}\right)_{n=0}^{\infty}$$

is non-increasing (obvious for $|z|<1$), and $\Delta_\infty(z)$ is a point if and only if this sequence tends to zero for $n\to\infty$.

In the remaining part of this paper we assume that

$$\sum_{k=1}^{\infty}(1-|\alpha_k|)<\infty. \tag{6.6}$$

Then the Blaschke product

$$\mathbb{B}(z)=\prod_{k=1}^{\infty}\frac{\overline{\alpha_k}}{|\alpha_k|}\,\frac{\alpha_k-z}{1-\overline{\alpha_k}z}$$

converges uniformly in every compact subset of $\mathbb{C}\backslash\{1/\overline{\alpha_j}: j=1,2,...\}$. The zeros of $\mathbb{B}$ are precisely $\alpha_1,\alpha_2,\alpha_3,....$ Notice that (6.6) implies that D_0 and E_0 are open sets in $\mathbb{C}$. Also we remark that for $z\in D_0\cup E_0$ now $\Delta_\infty(z)$ is a point if and only if

$$\sum_{k=0}^{\infty}|\phi_k(z)|^2=\infty.$$

PROPOSITION 6.1. *Let $z \in D_0 \cup E_0$ be given. Then*

(a) *The recurrency* (4.1) *has at least one solution* $(X_n, X_n^\dagger)$ *for which*

$$\sum_{k=0}^{\infty} |X_k|^2 < \infty, \qquad \textit{i.e.,} \quad (X_k)_{k=0}^{\infty} \in l^2.$$

(b) *For every solution* $(X_n, X_n^\dagger)$ *of* (4.1) *the sequence* $(X_k)_{k=0}^{\infty}$ *belongs to* l^2 *if and only if* $\Delta_\infty(z)$ *is a circular disk with a positive radius.*

Proof. (a) Take $X_n = \psi_n(z) - s\phi_n(z)$ and $X_n^\dagger = -\psi_n^*(z) - s\phi_n^*(z)$ with $s \in \Delta_\infty(z)$.

(b) If $\Delta_\infty(z)$ is not a single point, then $\sum_{k=0}^{\infty} |\phi_k(z)|^2 < \infty$ since the radius of $\Delta_\infty(z)$ is positive, and for $s \in \Delta_\infty(z)$ also $\sum_{k=0}^{\infty} |\psi_k(z) - s\phi_k(z)|^2 < \infty$. This implies that also $\sum_{k=0}^{\infty} |\psi_k(z)|^2 < \infty$. The first statement of (b) now follows from the fact that $(\phi_n(z), \phi_n^*(z))$ and $(\psi_n(z), -\psi_n^*(z))$ form a basis for the space of solutions of (4.1). Conversely, if $(X_n)_{n=0}^{\infty}$ is in l^2 for every solution $(X_n, X_n^\dagger)$ of (4.1), then $\sum_{k=0}^{\infty} |\phi_k(z)|^2 < \infty$, and therefore $\Delta_\infty(z)$ is a disk with a positive radius. ∎

In the sequel following we say that $\Delta_\infty(z)$ is a "disk" if $\Delta_\infty(z)$ is not a single point. Thus by a disk we mean a disk with a positive radius. Similarly we say that $K_\infty(z)$ is a "circle" if $K_\infty(z)$ does not reduce to a single point.

THEOREM 6.2 (Invariance). *Let $\sum_{k=1}^{\infty} (1 - |\alpha_k|) < \infty$ and let $z_0 \in D_0 \cup E_0$ be such that $\Delta_\infty(z_0)$ is a disk. Then $\Delta_\infty(z)$ is a disk for every $z \in D_0 \cup E_0$ and*

$$\sum_{k=0}^{\infty} |\phi_k(z)|^2 \qquad \textit{and} \qquad \sum_{k=0}^{\infty} |\psi_k(z)|^2$$

converge uniformly on every compact subset of $D_0 \cup E_0$.

For the proof of this theorem we need some consequences of the analogues of Green's formula and of the determinant formula.

From the Christoffel–Darboux formula (2.1) and its superstar conjugate (2.7) both with $z = w$, it follows that

$$\sum_{k=0}^{n} |\phi_k(z)|^2 = \sum_{k=0}^{n} |\mathbb{B}_{n\setminus k}(z)|^2 \, |\phi_k^*(z)|^2$$

for each n. Similarly (4.7) and (4.8) imply that

$$\sum_{k=0}^{n} |\psi_k(z)|^2 = \sum_{k=0}^{n} |\mathbb{B}_{n\setminus k}(z)|^2 \, |\psi_k^*(z)|^2$$

for each n. As for $z \in D_0 \cup E_0$ we have

$$0 < |\mathbb{B}(z)| < |\mathbb{B}_n(z)| \leqslant |\mathbb{B}_{n\backslash k}(z)| \leqslant 1, \qquad k = 0, 1, ..., n \qquad \text{if} \quad z \in D_0$$

and

$$1 \leqslant |\mathbb{B}_{n\backslash k}(z)| \leqslant |\mathbb{B}_n(z)| < |\mathbb{B}(z)| < \infty, \qquad k = 0, 1, ..., n \qquad \text{if} \quad z \in E_0,$$

we obtain

$$\sum_{k=0}^{\infty} |\phi_k(z)|^2 < \infty \Leftrightarrow \sum_{k=0}^{\infty} |\phi_k^*(z)|^2 < \infty \tag{6.7}$$

and

$$\sum_{k=0}^{\infty} |\psi_k(z)|^2 < \infty \Leftrightarrow \sum_{k=0}^{\infty} |\psi_k^*(z)|^2 < \infty \tag{6.8}$$

if $z \in D_0 \cup E_0$.

Next we consider (2.1) in the form

$$\phi_n^*(z)\, \overline{\phi_n^*(z_0)} - \phi_n(z)\, \overline{\phi_n(z_0)} = [1 - \zeta_n(z)\, \overline{\zeta_n(z_0)}] \sum_{k=0}^{n-1} \phi_k(z)\, \overline{\phi_k(z_0)}, \tag{6.9}$$

formula (4.6) in the form

$$\psi_n^*(z)\, \overline{\psi_n^*(z_0)} - \psi_n(z)\, \overline{\psi_n(z_0)} = [1 - \zeta_n(z)\, \overline{\zeta_n(z_0)}] \sum_{k=0}^{n-1} \psi_k(z)\, \overline{\psi_k(z_0)}, \tag{6.10}$$

and formula (4.5) in the forms

$$\begin{aligned} &-\phi_n^*(z)\, \overline{\psi_n^*(z_0)} - \phi_n(z)\, \overline{\psi_n(z_0)} \\ &\qquad = [1 - \zeta_n(z)\, \overline{\zeta_n(z_0)}] \left\{ \frac{2}{1 - z\overline{z_0}} + \sum_{k=0}^{n-1} \phi_k(z)\, \overline{\psi_k(z_0)} \right\} \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} &-\psi_n^*(z)\, \overline{\phi_n^*(z_0)} - \psi_n(z)\, \overline{\phi_n(z_0)} \\ &\qquad = [1 - \zeta_n(z)\, \overline{\zeta_n(z_0)}] \left\{ \frac{2}{1 - z\overline{z_0}} + \sum_{k=0}^{n-1} \psi_k(z)\, \overline{\phi_k(z_0)} \right\}. \end{aligned} \tag{6.12}$$

Elimination of $\phi_n^*(z)$ from (6.9) and (6.11) gives

$$\begin{aligned} &-[\overline{\phi_n(z_0)\, \psi_n^*(z_0)} + \overline{\psi_n(z_0)\, \phi_n^*(z_0)}]\, \phi_n(z) \\ &\qquad = [1 - \zeta_n(z)\, \overline{\zeta_n(z_0)}] \left\{ \frac{2}{1 - z\overline{z_0}}\, \overline{\phi_n^*(z_0)} \right. \\ &\qquad\qquad \left. + \sum_{k=0}^{n-1} [\overline{\phi_k(z_0)\, \psi_n^*(z_0)} + \overline{\psi_k(z_0)\, \phi_n^*(z_0)}]\, \phi_k(z) \right\}, \end{aligned} \tag{6.13}$$

while elimination of $\psi_n^*(z)$ from (6.10) and (6.12) leads to

$$
\begin{aligned}
&-[\overline{\phi_n(z_0)\,\psi_n^*(z_0)}+\overline{\psi_n(z_0)\,\phi_n^*(z_0)}]\,\psi_n(z)\\
&\quad=[1-\zeta_n(z)\,\overline{\zeta_n(z_0)}]\left\{\frac{2}{1-z\overline{z_0}}\,\overline{\psi_n^*(z_0)}\right.\\
&\qquad\left.+\sum_{k=0}^{n-1}[\overline{\phi_k(z_0)\,\psi_n^*(z_0)}+\overline{\psi_k(z_0)\,\phi_n^*(z_0)}]\,\psi_k(z)\right\}. \qquad (6.14)
\end{aligned}
$$

Using the analogue of the Liouville–Ostrogradskii formula (4.3) and

$$1-\zeta_n(z)\,\overline{\zeta_n(z_0)}=\frac{(1-|\alpha_n|^2)(1-z\overline{z_0})}{(1-\alpha_n\overline{z_0})(1-\overline{\alpha_n}z)}$$

we get

$$\frac{1-\zeta_n(z)\,\overline{\zeta_n(z_0)}}{\overline{\phi_n(z_0)\,\psi_n^*(z_0)}+\overline{\psi_n(z_0)\,\phi_n^*(z_0)}}=\frac{\overline{\alpha_n}-\overline{z_0}}{2\overline{z_0}\,\overline{\mathbb{B}_n(z_0)}}\,\frac{1-z\overline{z_0}}{1-\overline{\alpha_n}z}.$$

Thus (6.13) and (6.14) become

$$
\begin{aligned}
-\phi_n(z)=&\frac{\overline{\alpha_n}-\overline{z_0}}{\overline{z_0}\,\overline{\mathbb{B}_n(z_0)}}\,\frac{1}{1-\overline{\alpha_n}z}\,\overline{\phi_n^*(z_0)}\\
&+\frac{\overline{\alpha_n}-\overline{z_0}}{2\overline{z_0}\,\overline{\mathbb{B}_n(z_0)}}\,\frac{1-z\overline{z_0}}{1-\overline{\alpha_n}z}\sum_{k=0}^{n-1}[\overline{\phi_k(z_0)\,\psi_n^*(z_0)}+\overline{\psi_k(z_0)\,\phi_n^*(z_0)}]\,\phi_k(z),
\end{aligned}
\qquad (6.15)
$$

$$
\begin{aligned}
-\psi_n(z)=&\frac{\overline{\alpha_n}-\overline{z_0}}{\overline{z_0}\,\overline{\mathbb{B}_n(z_0)}}\,\frac{1}{1-\overline{\alpha_n}z}\,\overline{\psi_n^*(z_0)}\\
&+\frac{\overline{\alpha_n}-\overline{z_0}}{2\overline{z_0}\,\overline{\mathbb{B}_n(z_0)}}\,\frac{1-z\overline{z_0}}{1-\overline{\alpha_n}z}\sum_{k=0}^{n-1}[\overline{\phi_k(z_0)\,\psi_n^*(z_0)}+\overline{\psi_k(z_0)\,\phi_n^*(z_0)}]\,\psi_k(z).
\end{aligned}
\qquad (6.16)
$$

Proof of Theorem 6.2. In this proof we write

$$A_n(z)=-\frac{\overline{\alpha_n}-\overline{z_0}}{\overline{z_0}\,\overline{\mathbb{B}_n(z_0)}}\,\frac{1}{1-\overline{\alpha_n}z}\qquad\text{and}\qquad B_n(z)=-\frac{\overline{\alpha_n}-\overline{z_0}}{2\overline{z_0}\,\overline{\mathbb{B}_n(z_0)}}\,\frac{1-z\overline{z_0}}{1-\overline{\alpha_n}z}$$

and

$$a_{k,n}=\overline{\phi_k(z_0)\,\psi_n^*(z_0)}+\overline{\psi_k(z_0)\,\phi_n^*(z_0)}$$

for $k=0, 1, ..., n-1$ and $n=0, 1, ...$ Then

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}|a_{k,n}|^2 \leqslant 2\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}(|\phi_k(z_0)|^2\,|\psi_n^*(z_0)|^2+|\psi_k(z_0)|^2\,|\phi_n^*(z_0)|^2)$$

$$\leqslant 2\left(\sum_{n=0}^{\infty}|\psi_n^*(z_0)|^2\cdot\sum_{k=0}^{\infty}|\phi_k(z_0)|^2 + \sum_{n=0}^{\infty}|\phi_n^*(z_0)|^2\cdot\sum_{k=0}^{\infty}|\psi_k(z_0)|^2\right)<\infty$$

by (6.7) and (6.8) as $\sum_{k=0}^{\infty}|\phi_k(z_0)|^2<\infty$ and $\sum_{k=0}^{\infty}|\psi_k(z_0)|^2<\infty$ since $\Delta_\infty(z_0)$ is a disk. Let C be a compact subset of $D_0\cup E_0$. Then $A_n(z)$ and $B_n(z)$ are uniformly bounded for $z\in C$. Say $|A_n(z)|\leqslant R_1$ and $|B_n(z)|\leqslant R_2$ for $z\in C$ and $n=0, 1, 2, ...$ Then (6.15) and (6.16) are of the form

$$\eta_n=A_n c_n+B_n\sum_{k=0}^{n-1}a_{k,n}\eta_k, \qquad n=0, 1, 2, ...,$$

(with $\eta_n=\eta_n(z)$, $A_n=A_n(z)$, $B_n=B_n(z)$), where

$$\sum_{n=0}^{\infty}|c_n|^2<\infty \qquad \text{and} \qquad \sum_{n=0}^{\infty}\sum_{k=0}^{n-1}|a_{k,n}|^2<\infty.$$

As in Akhiezer's book [2] we show that $\sum_{n=0}^{\infty}|\eta_n|^2$ converges uniformly in C. Let $z\in C$. Then clearly

$$\left\{\sum_{n=m}^{N}|\eta_n|^2\right\}^{1/2}\leqslant R_1\left\{\sum_{n=m}^{N}|c_n|^2\right\}^{1/2}+R_2\left\{\sum_{n=m}^{N}\left|\sum_{k=0}^{n-1}a_{k,n}\eta_k\right|^2\right\}^{1/2}.$$

Let $0<\varepsilon<1$ and choose $m=m(\varepsilon, R_1, R_2)$ such that

$$\left\{\sum_{n=m}^{\infty}|c_n|^2\right\}^{1/2}<\frac{\varepsilon}{R_1} \qquad \text{and} \qquad \left\{\sum_{n=m}^{\infty}\sum_{k=0}^{n-1}|a_{k,n}|^2\right\}^{1/2}<\frac{\varepsilon}{R_2}.$$

Then for $N\geqslant m$ we have

$$\left\{\sum_{n=m}^{N}|\eta_n|^2\right\}^{1/2}\leqslant\varepsilon+R_2\left\{\sum_{n=m}^{N}\sum_{k=0}^{n-1}|a_{k,n}|^2\sum_{k=0}^{n-1}|\eta_k|^2\right\}^{1/2}$$

$$\leqslant\varepsilon+R_2\left\{\sum_{k=0}^{N}|\eta_k|^2\right\}^{1/2}\left\{\sum_{n=m}^{\infty}\sum_{k=0}^{n-1}|a_{k,n}|^2\right\}^{1/2}$$

$$\leqslant\varepsilon+\varepsilon\left\{\sum_{k=0}^{N}|\eta_k|^2\right\}^{1/2}$$

$$\leqslant\varepsilon+\varepsilon\left\{\sum_{k=m}^{N}|\eta_k|^2\right\}^{1/2}+\varepsilon\left\{\sum_{k=0}^{m-1}|\eta_k|^2\right\}^{1/2},$$

so

$$(1-\varepsilon)\left\{\sum_{n=m}^{N}|\eta_n|^2\right\}^{1/2}\leqslant\varepsilon+\varepsilon\left\{\sum_{k=0}^{m-1}|\eta_k|^2\right\}^{1/2}.$$

As $\sum_{k=0}^{m-1}|\eta_k|^2$ is continuous on C there is a constant $M>0$ such that

$$\left\{\sum_{k=0}^{m-1}|\eta_k|^2\right\}^{1/2}\leqslant M.$$

Hence

$$\left\{\sum_{n=m}^{N}|\eta_n|^2\right\}^{1/2}\leqslant\frac{\varepsilon(M+1)}{1-\varepsilon},\qquad\text{if}\quad N\geqslant m.$$

This implies that $\sum_{n=0}^{\infty}|\eta_n|^2$ converges uniformly in C. It follows from the above with $c_n=\overline{\phi_n^*(z_0)}$, $\eta_n=\phi_n(z)$ or $c_n=\overline{\psi_n^*(z_0)}$, $\eta_n=\psi_n(z)$ respectively that also $\sum_{n=0}^{\infty}|\phi_n(z)|^2<\infty$ and $\sum_{n=0}^{\infty}|\psi_n(z)|^2<\infty$ for $z\in C$, while both series converge uniformly in C.

In particular $\Delta_\infty(z)$ is a disk for each $z\in D_0\cup E_0$. ▮

We now may speak of an alternative:

either $\Delta_\infty(z)$ is a disk for every $z\in D_0\cup E_0$,
or $\Delta_\infty(z)$ is a point for every $z\in D_0\cup E_0$.

Thus we may say that Δ_∞ (or K_∞) is a disk (circle) or Δ_∞ (or K_∞) is a point without specifying any $z\in D_0\cup E_0$.

COROLLARY 6.3. *In the case of a limiting disk, the radius of $K_\infty(z)$ is a continuous function of z in $D_0\cup E_0$.*

Proof. Formula (6.4) and Theorem 6.2. ▮

THEOREM 6.4 (Analyticity). *Let Δ_∞ be a point. Then*

$$s(z)=\lim_{n\to\infty}R_n(z,w)$$

exists for $z\in D_0\cup E_0$ and $w\in T$ and is independent of w. The function s is analytic in $D_0\cup E_0$ and

$$\mathscr{R}\,\frac{s(z)}{1-|z|^2}>0\qquad(z\in D_0\cup E_0).\tag{6.17}$$

Proof. For $z \in D_0 \cup E_0$ let $s(z)$ be the point $\Delta_\infty(z)$. Then clearly $R_n(z, w) \to s(z)$ as $n \to \infty$, for each $w \in T$. Now take a fixed $w \in T$ and put $s_n(z) = R_n(z, w)$. Then s_n is analytic in $D_0 \cup E_0$. From the equation of $K_n(z)$ we obtain

$$\left| \frac{1}{\kappa_0} + s_n(z)\kappa_0 \right|^2 = |\psi_0 - s_n(z)\phi_0|^2 \leqslant 2\,\frac{s_n(z) + \overline{s_n(z)}}{1 - |z|^2},$$

so

$$\frac{1}{\kappa_0^2} + s_n(z) + \overline{s_n(z)} + |s_n(z)|^2\,\kappa_0^2 \leqslant 2\,\frac{s_n(z) + \overline{s_n(z)}}{1 - |z|^2}$$

which implies

$$|s_n(z)| \leqslant \frac{2}{\kappa_0^2}\,\frac{1 + |z|^2}{|1 - |z|^2|},$$

where the last member is uniformly bounded on compact subsets of $D_0 \cup E_0$. Thus $s(z)$ is analytic in $D_0 \cup E_0$. Finally (6.17) follows from the equation of $\Delta_\infty(z)$. ∎

7. THE MOMENT PROBLEM

Let $\mathscr{M}$ be the set of all the solutions of the moment problem mentioned in Section 1. Recall that a solution μ is a non-decreasing real valued function on $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi} R(t)\, d\mu(\theta) = M(R) \qquad \text{for} \quad R \in \mathscr{L} + \mathscr{L}_* \qquad (t = e^{i\theta}).$$

In Section 5 we already observed that $\mathscr{M} \neq \varnothing$. Two solutions μ_1 and μ_2 are considered to be equal if they determine the same continuous linear functional on $C(T)$, i.e., if

$$\int_{-\pi}^{\pi} f(t)\, d\mu_1(\theta) = \int_{-\pi}^{\pi} f(t)\, d\mu_2(\theta) \qquad \text{for all} \quad f \in C(T).$$

This means that μ_1 and μ_2 determine the same regular countably additive measure on the σ-field of the Borel sets in $[-\pi, \pi)$. This is just the case if there is a constant C such that $\mu_1(\theta) - \mu_2(\theta) = C$ at all θ where $\mu_1 - \mu_2$ is continuous. (Consider the Fourier series for $\mu_1 - \mu_2$.)

If $\mu \in \mathscr{M}$ then μ has infinitely many points of increase. (The corresponding Borel measure has infinite support.) Indeed, if μ has only finitely many points of increase, say $\theta_1, \theta_2, ..., \theta_n \in [-\pi, \pi)$, let $t_j = e^{i\theta_j}$ for $j = 1, ..., n$ and let $N(z) = (z - t_1)(z - t_2) \cdots (z - t_n)$ and consider

$$R(z) = \frac{N(z)}{\pi_n(z)} \in \mathscr{L}_n.$$

Clearly

$$R_*(z) = c\,\frac{N(z)}{\omega_n(z)} \in \mathscr{L}_{n*}$$

with $c \neq 0$. But then

$$0 < M(RR_*) = \int_{-\pi}^{\pi} |R(t)|^2 \, d\mu(\theta) = 0.$$

A contradiction.

For $\mu \in \mathscr{M}$ we define

$$F_\mu(z) = \int_{-\pi}^{\pi} \frac{t+z}{t-z}\, d\mu(\theta) \qquad (t = e^{i\theta}).$$

THEOREM 7.1. *For fixed $z \in D_0 \cup E_0$ we have*

$$\{F_\mu(z) \colon \mu \in \mathscr{M}\} = \Delta_\infty(z).$$

Proof. (a) Let $s = F_\mu(z)$ for some $\mu \in \mathscr{M}$. Put

$$f(t) = \frac{1 + \bar{z}t}{1 - \bar{z}t} \qquad \text{for} \quad t \in T.$$

Then

$$\overline{f(t)} = \frac{t+z}{t-z} = D(t, z), \qquad t \in T.$$

Let

$$\sum_{k=0}^{\infty} \gamma_k \phi_k$$

be the generalized Fourier series for f. Then

$$\gamma_k = \int_{-\pi}^{\pi} f(t)\, \overline{\phi_k(t)}\, d\mu(\theta), \qquad k = 0, 1, 2, ...,$$

so

$$\gamma_0 = \kappa_0 \int_{-\pi}^{\pi} f(t)\, d\mu(\theta) = \kappa_0 \bar{s}$$

and

$$\begin{aligned}\gamma_k &= \overline{\int_{-\pi}^{\pi} \frac{t+z}{t-z} (\phi_k(t) - \phi_k(z))\, d\mu(\theta)} + \overline{\int_{-\pi}^{\pi} \frac{t+z}{t-z} \phi_k(z)\, d\mu(\theta)} \\ &= \overline{-\psi_k(z) + s\phi_k(z)}, \qquad k = 1, 2, \ldots .\end{aligned}$$

Bessel's inequality gives

$$\sum_{k=0}^{\infty} |\gamma_k|^2 \leqslant \int_{-\pi}^{\pi} \left| \frac{t+z}{t-z} \right|^2 d\mu(\theta).$$

From

$$\left| \frac{t+z}{t-z} \right|^2 = -1 + \frac{1+|z|^2}{1-|z|^2} (f(t) + \overline{f(t)})$$

we get

$$\int_{-\pi}^{\pi} \left| \frac{t+z}{t-z} \right|^2 d\mu(\theta) = -\int_{-\pi}^{\pi} d\mu(\theta) + \frac{1+|z|^2}{1-|z|^2} (s+\bar{s}) = \frac{-1}{\kappa_0^2} - (s+\bar{s}) + \frac{2(s+\bar{s})}{1-|z|^2}.$$

Notice that $\phi_0(z) = \kappa_0 > 0$ and $\psi_0(z) = -1/\kappa_0$. Thus we also have

$$|\psi_0(z) - s\phi_0(z)|^2 = \left| \frac{1}{\kappa_0} + s\kappa_0 \right|^2 = \frac{1}{\kappa_0^2} + s + \bar{s} + |s|^2 \kappa_0^2 = \frac{1}{\kappa_0^2} + s + \bar{s} + |\gamma_0|^2.$$

Hence Bessel's inequality becomes

$$\sum_{k=0}^{\infty} |\psi_k(z) - s\phi_k(z)|^2 - \frac{1}{\kappa_0^2} - (s+\bar{s}) \leqslant \frac{-1}{\kappa_0^2} - (s+\bar{s}) + \frac{2(s+\bar{s})}{1-|z|^2},$$

so

$$\sum_{k=0}^{\infty} |\psi_k(z) - s\phi_k(z)|^2 \leqslant \frac{2(s+\bar{s})}{1-|z|^2},$$

which means that $s \in \Delta_\infty(z)$.

(b) Now let $s \in \Delta_\infty(z)$. Let us assume first that s is a boundery point of $\Delta_\infty(z)$. Then for each n there is a point $s_n \in K_n(z)$ such that $s_n \to s$ as $n \to \infty$. For each n there is a point $w_n \in T$ such that $R_n(z, w_n) = s_n$. Let μ_n be the solution of the truncated moment problem (in $\mathscr{L}_{(n-1)*} + \mathscr{L}_{(n-1)}$) with parameter w_n. Then

$$s_n = R_n(z, w_n) = \int_{-\pi}^{\pi} \frac{t+z}{t-z}\, d\mu_n(\theta).$$

By Helly's selection theorem there is a subsequence $(\mu_{n_j})_{j=1}^{\infty}$ of $(\mu_n)_{n=1}^{\infty}$ and a non-decreasing function μ on $[-\pi, \pi]$ such that $\mu_{n_j}(\theta) \to \mu(\theta)$ as $j \to \infty$ for all $\theta \in [-\pi, \pi]$. By Helly's convergence theorem

$$\int_{-\pi}^{\pi} g(\theta)\, d\mu_{n_j}(\theta) \to \int_{-\pi}^{\pi} g(\theta)\, d\mu(\theta) \qquad \text{as} \quad j \to \infty$$

for all continuous g on $[-\pi, \pi]$. Clearly $\mu \in \mathscr{M}$. Moreover

$$s_{n_j} = \int_{-\pi}^{\pi} \frac{t+z}{t-z}\, d\mu_{n_j}(\theta) \to \int_{-\pi}^{\pi} \frac{t+z}{t-z}\, d\mu(\theta) \qquad \text{as} \quad j \to \infty.$$

Since $s_n \to s$ for $n \to \infty$, this implies that

$$s = \int_{-\pi}^{\pi} \frac{t+z}{t-z}\, d\mu(\theta),$$

so $s = F_\mu(z)$ for some $\mu \in \mathscr{M}$.

Now assume that $\Delta_\infty(z)$ is a disk and that s belongs to the interior of $\Delta_\infty(z)$. Then s is a convex combination $\lambda s_1 + (1-\lambda) s_2$, $(0 < \lambda < 1)$ of points s_1, s_2 in the boundary of $\Delta_\infty(z)$. By the above there are $\mu_1, \mu_2 \in \mathscr{M}$ such that

$$s_j = \int_{-\pi}^{\pi} \frac{t+z}{t-z}\, d\mu_j(\theta), \qquad j = 1, 2.$$

Clearly $\mu = \lambda \mu_1 + (1-\lambda) \mu_2 \in \mathscr{M}$ and $s = F_\mu(z)$.

COROLLARY 7.2. *In the case of a limiting disk, for each $s \in \Delta_\infty(z)$ $(z \in D_0 \cup E_0)$ there is a $\mu \in \mathscr{M}$ such that $s = F_\mu(z)$. In this case the moment problem has more than one solution.*

COROLLARY 7.3. *In the case of a limiting point the moment problem has a unique solution.*

Proof. If $\mu_1, \mu_2 \in \mathscr{M}$, then the functions F_{μ_1} and F_{μ_2} coincide on $\mathbb{C}\backslash T$. For $\mu = \mu_1 - \mu_2$ we have

$$\int_{-\pi}^{\pi} \frac{t+z}{t-z}\, d\mu(\theta) = 0 \qquad \text{for} \quad z \in \mathbb{C}\backslash T,$$

while μ is of bounded variation on $[-\pi, \pi]$. Considering the power series of the function

$$F(z) = \int_{-\pi}^{\pi} \frac{t+z}{t-z}\, d\mu(\theta)$$

around 0 and around ∞ we see that

$$\int_{-\pi}^{\pi} t^k\, d\mu(\theta) = 0 \qquad \text{for} \quad k \in \mathbb{Z}.$$

It follows by integration by parts that

$$0 = \int_{-\pi}^{\pi} e^{ik\theta}\, d\mu(\theta) = e^{ik\theta}\mu(\theta)|_{-\pi}^{\pi} - ik \int_{-\pi}^{\pi} e^{ik\theta}\mu(\theta)\, d\theta$$

$$= -ik \int_{-\pi}^{\pi} e^{ik\theta}\mu(\theta)\, d\theta, \qquad k \in \mathbb{Z}.$$

This implies that all the Fourier coefficients of μ, except possibly the zeroth coefficient, are zero. Thus there is a constant C such that $\mu(\theta) = C$ at all the points where μ is continuous. Hence $\mu_1 = \mu_2$. ∎

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